

## MULTIGRADED APOLARITY

MACIEJ GAŁĄZKA

ABSTRACT. We generalize methods to compute various kinds of rank to the case of a toric variety  $X$  embedded into projective space using a very ample line bundle  $\mathcal{L}$ . We use this to compute rank, border rank, and cactus rank of monomials in  $H^0(X, \mathcal{L})^*$  when  $X$  is the Hirzebruch surface  $\mathbb{F}_1$ , the weighted projective plane  $\mathbb{P}(1, 1, 4)$ , or a fake projective plane.

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*Date:* January 26, 2016.

*2010 Mathematics Subject Classification.* 14M25, 14N15.

*Key words and phrases.* secant variety, Waring rank, cactus rank, border rank, toric variety, apolarity, catalecticant.

## 1. INTRODUCTION

## 1.1. Preliminaries.

**Definition 1.1.** Let  $W$  be a finite-dimensional complex vector space, and  $X$  a subvariety of  $\mathbb{P}W$ . Let

$$\sigma_r^0(X) = \{[F] \in \mathbb{P}W : [F] \in \langle p_1, \dots, p_r \rangle \text{ where } p_1, \dots, p_r \in X\},$$

where  $\langle \rangle$  denotes the (projective) linear span. Define the  $r$ -th *secant variety* of  $X \subseteq \mathbb{P}W$  by  $\sigma_r(X) = \overline{\sigma_r^0(X)}$ . The overline denotes the Zariski closure. For any non-zero  $F \in W$  define the  $X$ -rank of  $F$ :

$$\begin{aligned} r_X(F) &= \min\{r \in \mathbb{Z}_{\geq 1} : [F] \in \sigma_r^0(X)\} \\ &= \min\{r \in \mathbb{Z}_{\geq 1} : [F] \in \langle p_1, \dots, p_r \rangle \text{ for some } p_1, \dots, p_r \in X\} \end{aligned}$$

and the  $X$ -border rank of  $F$ :

$$\begin{aligned} \mathbf{r}_X(F) &= \min\{r \in \mathbb{Z}_{\geq 1} : [F] \in \sigma_r(X)\} \\ &= \min\{r \in \mathbb{Z}_{\geq 1} : F \text{ is a limit of points of } X\text{-rank} \leq r\}. \end{aligned}$$

Usually, if  $X$  is fixed, we omit the prefix and call them rank and border rank, respectively.

The problem of calculating border rank of points is related to finding equations of secant varieties. Namely, if we know set-theoretic equations of  $\sigma_r(X) \subseteq \mathbb{P}W$  for  $r = 1, 2, 3, \dots$ , then we can calculate the border rank of any point (by checking if it satisfies the equations).

An important special case is when  $X$  is the  $d$ -th Veronese variety  $\mathbb{P}V \subseteq \mathbb{P}\text{Sym}^d V$ . Then  $X$ -rank of  $[F] \in \mathbb{P}\text{Sym}^d V$  is the least  $r$  such that  $F$  can be written as  $v_1^d + \dots + v_r^d$  for some  $v_i \in V$ . The  $X$ -rank is called the symmetric rank, or the Waring rank in this case.

In this case we may use apolarity action to help us to calculate rank. Let us define it. Let  $\text{Sym}^\bullet V^*$  be the coordinate ring of  $\mathbb{P}V$ . Fix a basis  $y_0, \dots, y_n$  of  $V$ . Let  $x_0, \dots, x_n$  be the dual basis of  $V^*$ . We consider the action of  $\text{Sym}^\bullet V^* = \mathbb{C}[x_0, \dots, x_n]$  on  $\text{Sym}^\bullet V = \mathbb{C}[y_0, \dots, y_n]$  denoted by  $\circ$  and induced by

$$(1) \quad x_i \circ y_0^{b_0} \cdot \dots \cdot y_n^{b_n} = \begin{cases} b_i \cdot y_0^{b_0} \cdot \dots \cdot y_i^{b_i-1} \cdot \dots \cdot y_n^{b_n} & \text{if } b_i > 0, \\ 0 & \text{otherwise} \end{cases}$$

(so it can be seen as derivation).

This makes  $\mathbb{C}[y_0, \dots, y_n]$  into a  $\mathbb{C}[x_0, \dots, x_n]$ -module.

Let  $\mathbb{C}[y_0, \dots, y_n]_d := \text{Sym}^d V$  denote the  $d$ -th graded piece of  $\mathbb{C}[y_0, \dots, y_n]$  (where  $d \in \mathbb{Z}$ ). For  $F \in \mathbb{C}[y_0, \dots, y_n]_d$ , let  $F^\perp$  denote its annihilator, which is a homogeneous ideal of  $\mathbb{C}[x_0, \dots, x_n]$ .

The apolarity action can help us to calculate rank by means of the following proposition.

**Proposition 1.2** (Classical Apolarity Lemma). *Let  $F \in \mathbb{C}[y_0, \dots, y_n]_d$  for some integer  $d \geq 1$ . For any closed subset  $Y \subseteq \mathbb{P}(\mathbb{C}[y_0, \dots, y_n]_1)$  let  $I(Y)$*

denote its vanishing ideal in  $\mathbb{C}[x_0, \dots, x_n]$ . Then for any set of one-forms  $Z = \{l_1, \dots, l_r\} \subseteq \mathbb{C}[y_0, \dots, y_0]_1$  we have

$$F \in \langle l_1^d, \dots, l_r^d \rangle \iff I(Z) \subseteq F^\perp.$$

To see how this works, let us look at the following

**Example 1.3.** Look at the  $d$ -th Veronese embedding  $\mathbb{C}[y_0, y_1]_1 \hookrightarrow \mathbb{C}[y_0, y_1]_d$  (where  $d \geq 2$ ), and let  $F = y_0^{d-1}y_1$ . Here  $F^\perp = (x_0^d, x_1^2)$ . Then  $I = (x_0^d - x_1^d) \subset F^\perp$  is a radical ideal of  $d$  points, so  $r(F) \leq d$ . For a proof that  $r(F) = d$ , see [IK99, Theorem 1.44] (an algorithm for computing the rank of any form in two variables) or [CCG12] (where the authors determine the rank of any monomial).

**1.2. Background.** The topic goes back to works of Sylvester on apolarity in the 19th century. For introductions to this subject, see [Lan12] and [IK99]. For a short introduction to the concept of rank for many different subvarieties  $X \subseteq \mathbb{P}^N$  and many ways to give lower bounds for rank, see [Tei14] (see also many references there). For a short review of the apolarity action in the case of the Veronese embedding, see [BB14, Section 3].

In this paper, we see what happens when  $X$  is a toric variety. For an introduction to this subject, see the newer [CLS11] and the older [Ful93]. For toric varieties, many invariants can be computed quite easily. This can be used to study ranks and secant varieties. In [CS07], the authors investigate the second secant variety  $\sigma_2(X)$ , where  $X$  is a toric variety embedded into some projective space. As they write there, “Many classical varieties whose secant varieties have been studied are toric”. Here we take a different approach. We generalize apolarity to toric varieties, and then, as an application, we compute rank, cactus rank and border rank of some polynomials.

**1.3. Cactus rank.** For a zero-dimensional scheme  $R$  (of finite type over  $\mathbb{C}$ ), let  $\text{length } R$  denote its length, i.e.  $\dim_{\mathbb{C}} H^0(R, \mathcal{O}_R)$ . This is equal to the degree of  $R$  in any embedding into projective space. Also for any subscheme  $R \hookrightarrow \mathbb{P}^N$  define  $\langle R \rangle$  to be the linear span of  $R$ , i.e. the smallest projective linear space, through which the inclusion of the scheme factors.

**Definition 1.4.** Define the  $X$ -cactus rank:

$$\text{cr}_X(F) = \min\{\text{length } R : R \hookrightarrow X, \dim R = 0, F \in \langle R \rangle\}.$$

We have the following inequalities:

$$\begin{aligned} \text{cr}(F) &\leq r(F), \\ \underline{\text{r}}(F) &\leq r(F). \end{aligned}$$

As far we know, the notion of cactus rank was first defined in [IK99, Chapter 5] (where it is called the “scheme length”). For motivation, basic properties and application in the case of the Veronese embedding, see [BB14]. One of the main reasons to study cactus rank is the fact that properties of the Hilbert scheme of all zero-dimensional subschemes of a variety are better

understood than properties of the subset corresponding to smooth schemes (i.e. schemes of points). Another reason is the fact that many bounds for rank work also for cactus rank, for instance the Landsberg-Ottaviani bound for vector bundles (see [Gal]). There is also a lower bound for the cactus rank by Ranestad and Schreyer (see [RS11]).

Suppose we go back to the case of the Veronese embedding. Then we can make Proposition 1.2 work for the cactus rank:

**Proposition 1.5.** *Let  $F \in \mathbb{C}[y_0, \dots, y_n]_d$ . Then for any zero-dimensional closed subscheme*

$$R \hookrightarrow \mathbb{P}(\mathbb{C}[y_0, \dots, y_n]_1)$$

*we have*

$$F \in \langle R \rangle \iff I(R) \subseteq F^\perp,$$

*where  $I(R)$  denotes the saturated ideal of  $R$  in  $\mathbb{C}[x_0, \dots, x_n]$ .*

Let us go back to Example 1.3. We can see that  $(x_1^2) \subseteq F^\perp$ , and the length of the subscheme defined by  $x_1^2$  is two. This means that the cactus rank is at most two. In fact, it is two, because  $y_0^{d-1}y_1$  is not a  $d$ -th power.

**1.4. Main results.** We need to introduce some notions to state the main results. We revisit these ideas in a different language in the following sections. Suppose  $X$  is a smooth projective toric variety. Then  $\mathrm{Cl} X = \mathrm{Pic} X$  is free of finite rank. Fix a basis  $\mathcal{L}_1, \dots, \mathcal{L}_l$  of  $\mathrm{Cl} X$ , where  $\mathcal{L}_i$  are line bundles. Define the Cox ring of  $X$ :

$$S = \bigoplus_{m_1, \dots, m_l \in \mathbb{Z}} H^0(X, \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_l^{\otimes m_l}),$$

where the multiplication is the tensor product of sections. By definition, it is graded by  $\mathrm{Cl} X$ . Since  $X$  is a toric variety,  $S$  is a polynomial ring with finitely many variables (see [CLS11, Section 5.2]), so we may write  $S \cong \mathbb{C}[x_1, \dots, x_r]$ . We look at the ring  $T = \mathbb{C}[y_1, \dots, y_r]$ . We forget that it is a ring, and treat it as an  $S$ -module with action  $\circ$  defined as in Equation (1). We define a grading on  $T$  in  $\mathrm{Cl}(X)$  in an analogous way as on  $S$ :

$$\deg y_1^{a_1} \cdot \dots \cdot y_r^{a_r} = \deg x_1^{a_1} \cdot \dots \cdot x_r^{a_r}.$$

Then we identify  $H^0(X, \mathcal{L})^*$  with the graded piece of  $T$  of degree  $\mathcal{L}$  (this identification is described in Proposition 3.3 in detail), and for a homogeneous  $F \in T$  we define  $F^\perp$  as its annihilator in  $S$  (with respect to the action  $\circ$ ).

Let  $R \hookrightarrow X$  be any closed subscheme with ideal sheaf  $\mathcal{I}_R$ . Then we can define  $I(R)$ , the ideal of  $R$  in  $S$ , by

$$I(R) = \bigoplus_{m_1, \dots, m_l \in \mathbb{Z}} H^0(X, \mathcal{I}_R \otimes \mathcal{L}_1^{\otimes m_1} \otimes \dots \otimes \mathcal{L}_l^{\otimes m_l}).$$

Recall that for any line bundle  $\mathcal{L}$  the space  $H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \subseteq H^0(X, \mathcal{L})$  is the subspace of those sections which are zero when pulled back to  $R$ .

The main result of this paper is:

**Theorem 1.6** (Multigraded Apolarity Lemma). *Let  $X$  be a smooth toric variety. Let  $\mathcal{L}$  be a very ample line bundle on  $X$  and consider the associated morphism  $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$ . Take any  $F \in H^0(X, \mathcal{L})^* = T_{\mathcal{L}}$ . Then for any subscheme  $R \hookrightarrow X$ .*

$$I(R) \subseteq F^\perp \iff F \in \langle R \rangle.$$

Here  $\langle R \rangle$  denotes the linear span of  $R$  in  $H^0(X, \mathcal{L})^*$ .

This was independently proven (even for general varieties) in [GRV16, Lemma 1.3]. In the paper the authors use this to determine varieties of apolar subschemes for  $\mathbb{P}^1 \times \mathbb{P}^1$  embedded into projective space by  $\mathcal{O}(2, 2)$  and  $\mathcal{O}(3, 3)$ , and also for the Hirzebruch surface  $\mathbb{F}_1$  embedded by the bundle  $\mathcal{O}(2, 1)$  (in notation from Subsection 5.1).

Theorem 1.6 allows us to give upper bounds for cactus rank (when  $R$  is a zero-dimensional subscheme) and rank (when  $R$  is a reduced zero-dimensional subscheme).

We prove Theorem 1.6 in Section 3.2 (Proposition 3.8) in a more general form: we allow  $X$  to be a  $\mathbb{Q}$ -factorial projective toric variety (so that  $\text{Pic } X$  has finite index in  $\text{Cl } X$ ), in the toric language the corresponding notion is a simplicial fan. Then it may happen that the class group has torsion.

We may think of it from a different perspective: start with  $T = \mathbb{C}[y_1, \dots, y_r]$ , pick a finitely generated abelian group  $G$ , and assign to each  $i$  ( $1 \leq i \leq r$ ) an element  $g_i \in G$ , and then we define the grading on  $T$  in  $G$  by

$$\deg y_i = g_i.$$

We define the grading on  $S$  in an analogous way. Then we choose a homogeneous polynomial  $F \in T_g$  and look for a toric variety  $X$  such that  $G$  is the class group of  $X$  and  $g$  is a very ample class.

To sum up, we look at the polynomial ring  $\mathbb{C}[y_1, \dots, y_r]$  with a grading in a finitely generated group  $G$  (such that monomials are homogeneous), and we generalize the apolarity action to this multigraded setting.

From it we derive a corollary.

Let  $X$  be a projective  $\mathbb{Q}$ -factorial toric variety. Suppose  $\mathcal{M}$  is a reflexive sheaf of rank one on  $X$  (which corresponds to the class of a Weil divisor). Let  $\mathcal{L}$  be a very ample line bundle on  $X$  giving the embedding  $X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$ . Consider the restriction of the action  $\circ$  to

$$S_{\mathcal{M}} \times T_{\mathcal{L}} \xrightarrow{\circ} T_{\mathcal{L} \otimes \mathcal{M}^\vee}.$$

For any  $F \in T_{\mathcal{L}}$  we consider the linear map  $C_F^{\mathcal{M}} : S_{\mathcal{M}} \rightarrow T_{\mathcal{L} \otimes \mathcal{M}^\vee}$  given by  $h \mapsto h \circ F$ .

**Proposition 1.7.** *Fix  $F \in H^0(X, \mathcal{L})^*$ . We have the following:*

(1) *if  $\mathcal{M}$  is a reflexive sheaf of rank one, then*

$$\mathfrak{r}(F) \geq \text{rank}(C_F^{\mathcal{M}}).$$

(2) if  $\mathcal{M}$  is a line bundle, then

$$\mathrm{cr}(F) \geq \mathrm{rank}(C_F^{\mathcal{M}}).$$

We also provide an example such that the bound in point (1) does not hold for cactus rank, see Remark 5.4.

Proposition 1.7 is proven in Corollary 4.2 and Corollary 4.5. The bound in point (2) was given in [IK99, Theorem 5.3.D] for the Veronese embedding. Also see [Gal] for a version of the bound in point (2) for vector bundles of higher rank.

Finally, we use this to compute ranks of monomials when  $X$  is a projective toric surface. The first example is the Hirzebruch surface  $\mathbb{F}_1$  (which can be defined as  $\mathbb{P}^2$  blown up in one point), see Subsection 5.1 Here we find monomials whose border rank is less than their cactus rank (and also their smoothable rank), see Remark 5.2. The next one is the weighted projective plane  $\mathbb{P}(1, 1, 4)$ , and the last one is a fake projective plane — the quotient of  $\mathbb{P}^2$  by the action of  $\mathbb{Z}/3 = \{1, \varepsilon, \varepsilon^2\}$  (where  $\varepsilon^3 = 1$ ) given by  $\varepsilon \cdot [\lambda_0, \lambda_1, \lambda_2] = [\lambda_0, \varepsilon\lambda_1, \varepsilon^2\lambda_2]$ . See Subsections 5.2 and 5.3 respectively. Here we give examples of monomials whose cactus rank is less than their border rank.

**1.5. Acknowledgments.** This article is based on my master thesis, [Gal14].

I thank my advisor, Jarosław Buczyński, for introducing me to this subject, his insight, many suggestions of examples, suggestions on how to improve the presentation, many discussions, and constant support. I also thank Piotr Achinger and Joachim Jelisiejew on suggestions how to improve the presentation. I was supported by the project “Secant varieties, computational complexity, and toric degenerations” realized withing the Homing Plus programme of Foundation for Polish Science, co-financed from European Union, Regional Development Fund, and by Warsaw Center of Mathematics and Computer Science financed by Polish program KNOW.

**1.6. Secant varieties.** In the remainder of this introduction we review general facts about secant varieties.

Let  $X \subseteq \mathbb{P}W$ . Recall the definitions of secant varieties of  $X \subseteq \mathbb{P}W$  and of the rank of  $F \in \mathbb{P}W$  from Definition 1.1.

**Proposition 1.8** (Terracini’s Lemma). *Let  $r$  be a positive integer. Then for  $r$  general points  $p_1, \dots, p_r \in X$  and a general point  $q \in \langle p_1, \dots, p_r \rangle$  we have*

$$\mathbb{P}T_q\sigma_r(X) = \langle \mathbb{P}T_{p_1}X, \dots, \mathbb{P}T_{p_r}X \rangle.$$

Here  $\mathbb{P}T_qX$  denotes the projective tangent space of  $X$  embedded in  $\mathbb{P}W$  at point  $q$ , i.e. the projectivization of the affine tangent space to the affine cone of  $X$ .

For a proof, see [Lan12, Section 5.3] or [Zak93, Chapter V, Proposition 1.4].

**Corollary 1.9** (of Proposition 1.8). *The dimension of  $\sigma_r(X)$  is not greater than  $r(\dim X + 1) - 1$ .*

**Proposition 1.10.** *If  $X$  is irreducible, then  $\sigma_r(X)$  is irreducible for any  $r \geq 1$ .*

**Definition 1.11.** When  $\dim \sigma_r(X) = \min(\dim \mathbb{P}W, r(\dim X + 1) - 1)$ , we say that  $\sigma_r(X)$  is of *expected dimension*.

## 2. TORIC VARIETIES

**2.1. Quotient construction and the Cox ring.** Let  $M$  and  $N$  be dual lattices (abelian groups isomorphic to  $\mathbb{Z}^k$  for some  $k \geq 1$ ) and  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$  be the duality between them. Let  $X_\Sigma$  be the toric variety of a fan  $\Sigma \subseteq N_\mathbb{R} := N \otimes \mathbb{R}$  with no torus factors. The term “with no torus factors” means that the linear span of  $\Sigma$  in  $N_\mathbb{R}$  is the whole space. Let  $\Sigma(1)$  denote the set of rays of the fan  $\Sigma$ . Similarly  $\sigma(1)$  denotes the set of rays in the cone  $\sigma$ . Then  $X_\Sigma$  can be obtained as an almost geometric quotient of an action of  $G := \text{Hom}(\text{Cl } X_\Sigma, \mathbb{C}^*)$  on  $\mathbb{C}^{\Sigma(1)} \setminus Z$ , where  $Z$  is a subvariety of  $\mathbb{C}^{\Sigma(1)}$ . Let us go briefly through the construction of this quotient. We follow [CLS11, Section 5.1].

Since  $X_\Sigma$  has no torus factors, we have an exact sequence

$$0 \rightarrow M \rightarrow \mathbb{Z}^{\Sigma(1)} \rightarrow \text{Cl } X_\Sigma \rightarrow 0.$$

After applying  $\text{Hom}(-, \mathbb{C}^*)$ , this gives

$$0 \rightarrow \text{Hom}(\text{Cl } X_\Sigma, \mathbb{C}^*) \rightarrow (\mathbb{C}^*)^{\Sigma(1)} \rightarrow T_N \rightarrow 0,$$

where  $T_N = \mathbb{C}^* \otimes N \subseteq X_\Sigma$  is the torus of  $X_\Sigma$ . So  $G = \text{Hom}(\text{Cl } X_\Sigma, \mathbb{C}^*)$  is a subset of  $\mathbb{C}^{\Sigma(1)}$ , and the action is given by multiplication on coordinates. Let  $S = \text{Spec } \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ . In other words,  $S$  is the polynomial ring with variables indexed by the rays of the fan  $\Sigma$ . The ring  $S$  is the coordinate ring of the affine space  $\mathbb{C}^{\Sigma(1)}$ . For a cone  $\sigma \in \Sigma$ , define

$$x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho.$$

Then define a homogeneous ideal in  $S$ :

$$(2) \quad B = B(\Sigma) = (x^{\hat{\sigma}} : \sigma \in \Sigma) \subseteq S,$$

which is called the irrelevant ideal, and let  $Z = Z(\Sigma) \subseteq \mathbb{C}^{\Sigma(1)}$  be the vanishing set of  $B$ . For a precise construction of the quotient map  $\pi$

$$\mathbb{C}^{\Sigma(1)} \setminus Z \xrightarrow{\pi} (\mathbb{C}^{\Sigma(1)} \setminus Z) // G = X_\Sigma$$

see [CLS11, Proposition 5.1.9]. If  $(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r = \mathbb{C}^{\Sigma(1)}$ , we sometimes write  $[\lambda_1, \dots, \lambda_r]$  for  $\pi(\lambda_1, \dots, \lambda_r)$ .

Fix an ordering of all the rays of the fan, let  $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ . Then  $S$  becomes  $\mathbb{C}[x_{\rho_1}, \dots, x_{\rho_r}] =: \mathbb{C}[x_1, \dots, x_r]$ . The ring  $S$  is the Cox ring of  $X_\Sigma$ .

For more details, see [CLS11, 5.2], where  $S$  is called the total coordinate ring. This ring is graded by the class group  $\text{Cl } X_\Sigma$ , where

$$\deg x_i = [D_{\rho_i}],$$

and  $D_{\rho_i}$  is the torus-invariant divisor corresponding to  $\rho_i$ , see [CLS11, Chapter 4]. For the rest of the paper, if  $A$  is a graded ring,  $A_\alpha$  will denote the graded piece of  $A$  of degree  $\alpha$ .

**2.2. Saturated ideals.** Take any ideals  $I, J \subseteq S$ . Let  $(I :_S J)$  be the set of all  $x \in S$  such that  $x \cdot J \subseteq I$ ; it is an ideal of  $S$ . It is sometimes called the quotient ideal, or the colon ideal. For any ideals  $I, J, K \subseteq S$  we have:

- $I \subseteq (I :_S J)$ ,
- if  $J \subseteq K$ , then  $(I :_S J) \supseteq (I :_S K)$ ,
- $(I :_S J \cdot K) = ((I :_S J) :_S K)$ .

Recall the irrelevant ideal  $B \subseteq S$  defined in Equation (2). Take any ideal  $I \subseteq S$ . We define the  $B$ -saturation of  $I$  as

$$I^{\text{sat}} = \bigcup_{i \geq 1} (I :_S B^i).$$

Note that this is an increasing union because  $B^i \supseteq B^j$  for  $i < j$ , so  $I^{\text{sat}}$  is an ideal. Since  $S$  is Noetherian, the union stabilizes in a finite number of steps. We always have  $I \subseteq I^{\text{sat}}$ . If this is an equality, we say that  $I$  is  $B$ -saturated. In order to show that  $I$  is  $B$ -saturated, it suffices to find any  $i \geq 1$  such that  $I = (I :_S B^i)$ .

Moreover, if  $I$  and  $J$  are homogeneous, then so is  $(I :_S J)$ . It follows that for  $I$  homogeneous the ideal  $I^{\text{sat}}$  is homogeneous.

**Example 2.1.** Let us look at the projective space  $\mathbb{P}_{\mathbb{C}}^k$ . See [CLS11, Example 5.1.7]. Here  $S = \mathbb{C}[x_0, \dots, x_k]$ ,  $B = (x_0, \dots, x_k) = \bigoplus_{i \geq 1} S_i$  and  $Z = \{0\}$ . In this case

$$I^{\text{sat}} = \{f \in S : \text{for all } i = 0, 1, \dots, k \text{ there is } n \text{ such that } x_i^n \cdot f \in I\}.$$

Recall that in this case there is a 1-1 correspondence between closed subschemes of  $\mathbb{P}_{\mathbb{C}}^k$  and homogeneous  $B$ -saturated ideals of  $S$ . Moreover, the ideal given by  $\bigoplus_{i \geq 0} H^0(X, \mathcal{I}_R \otimes \mathcal{O}(i))$ , where  $\mathcal{I}_R$  is the ideal sheaf of  $R$  in  $\mathbb{P}_{\mathbb{C}}^k$ , is  $B$ -saturated. For more on this, see [Har77, II, Corollary 5.16 and Exercise 5.10].

For a toric variety the situation is more complicated. We will assume that the fan  $\Sigma$  is simplicial for technical reasons. There can be many  $B$ -saturated ideals defining a subscheme  $R$ . But they have to agree in the Pic part. See [Cox95, Theorem 3.7 and the following discussion] for more details. Consider the map

$$(3) \quad \bigoplus_{\alpha \in \text{Cl } X_\Sigma} H^0(X, \mathcal{I}_R \otimes \mathcal{O}(\alpha)) \rightarrow \bigoplus_{\alpha \in \text{Cl } X_\Sigma} H^0(X, \mathcal{O}(\alpha))$$



induced by  $\mathcal{I}_R \hookrightarrow \mathcal{O}_{X_\Sigma}$ . We may take  $I$  to be the image of this map. This is done in the proof of [CLS11, Proposition 6.A.6]. Note that in this case for any  $\alpha \in \text{Pic } X_\Sigma$  the vector space  $H^0(X_\Sigma, \mathcal{I}_R \otimes \mathcal{O}(\alpha))$  can be identified with those global sections of  $\mathcal{O}(\alpha)$  which vanish on  $R$ . So let us make the following

**Definition 2.2.** Let  $X_\Sigma$  be a simplicial toric variety. Let  $R \hookrightarrow X_\Sigma$  be a closed subscheme. We define  $I(R) \subseteq S$ , the ideal of  $R$ , to be the image of homomorphism (3).

**Proposition 2.3.** Suppose the fan  $\Sigma$  is simplicial. Let  $\alpha \in \text{Pic } X_\Sigma$  be the class of a Cartier divisor. Let  $R \hookrightarrow X_\Sigma$  be any closed subscheme. Then  $(I(R))_\alpha = (I(R) :_S B)_\alpha$  (i.e.  $I(R)$  agrees with  $I(R)^{\text{sat}}$  in degree  $\alpha$ ).

*Proof.* Take  $x \in S_\alpha$  such that  $x \cdot B \subseteq I(R)$ . It is enough to show that  $x$  is zero on  $R$ . Take any point  $p \in R$ . We will show that  $x$  is zero on  $R$  around that point. Since the vanishing set of  $B$  is empty, we know that some homogeneous element  $b \in B$  is non-zero at  $p$ . By taking a big enough power, we may assume  $b \in B_\beta$  for some  $\beta \in \text{Pic } X_\Sigma$  (here we are using that  $\Sigma$  is simplicial!). Because  $b$  is non-zero at  $p$ , there is an open neighbourhood  $p \in U \subseteq X_\Sigma$  such that  $\mathcal{O}_{X_\Sigma}(\beta)$  is trivialized on  $U$  by  $b$ . But then  $x$  is zero when pulled back to  $R$  on  $U$  if and only if  $x \cdot b$  is zero when pulled back to  $R$  on  $U$ . But the latter thing is true as  $x \cdot b \in I(R)$ .  $\square$

**2.3. Isomorphism between sections and polynomials.** Let  $\alpha \in \text{Cl } X_\Sigma$ . Recall the isomorphism of  $H^0(X_\Sigma, \mathcal{O}(\alpha))$  and  $\mathbb{C}[x_1, \dots, x_r]_\alpha$  given in [CLS11, Proposition 5.3.7].

**Proposition 2.4.** Suppose  $\alpha \in \text{Pic } X_\Sigma$ . Take any section  $s \in H^0(X_\Sigma, \mathcal{O}(\alpha))$  and the corresponding polynomial  $f \in S_\alpha$ . Also let  $p$  be a point in  $X_\Sigma$  and take any  $(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$  such that  $\pi(\lambda_1, \dots, \lambda_r) = p$ . Then

$$s(p) = 0 \iff f(\lambda_1, \dots, \lambda_r) = 0.$$

*Proof.* Take any  $\sigma$  such that  $p \in U_\sigma$ . We will trivialize the line bundle  $\mathcal{O}(\alpha)$  on  $U_\sigma$  in order to move the situation to regular functions on  $U_\sigma$ . We will do it by finding a section that is nowhere zero both as a polynomial and as a section.

We know that  $U_\sigma = \text{Spec}(S_{x^{\hat{\sigma}}})_0$ , where  $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$ , the inner subscript refers to localization, and the outer one is taking degree 0. From the definition of  $\mathcal{O}(\alpha)$  we have  $H^0(U_\sigma, \mathcal{O}(\alpha)) = (S_{x^{\hat{\sigma}}})_\alpha$ . Our goal is to find a monomial in  $(S_{x^{\hat{\sigma}}})_\alpha$  which is nowhere zero as a section. Take any torus-invariant representative  $\sum_\rho a_\rho D_\rho$  of class  $\alpha$  (here  $a_\rho \in \mathbb{Z}$ ). From [CLS11, Theorem 4.2.8] there exists an  $m_\sigma \in M$  such that  $\langle m_\sigma, u_\rho \rangle = -a_\rho$  for  $\rho \in \sigma(1)$  (here  $\sigma(1)$  is the set of rays of the cone  $\sigma$ , and  $u_\sigma \in N$  is the generator of ray  $\rho$ ). Then

$$\sum_\rho \langle m_\sigma, u_\rho \rangle D_\rho + \sum_\rho a_\rho D_\rho = \sum_{\rho \notin \sigma(1)} (\langle m_\sigma, u_\rho \rangle + a_\rho) D_\rho$$

belongs to the class  $\alpha$  as well. This is a direct consequence of the exact sequence [CLS11, Theorem 4.2.1]. The outcome is that the monomial  $g := \prod_{\rho \notin \sigma(1)} x_\rho^{\langle m_\sigma, u_\rho \rangle + a_\rho}$  has degree  $\alpha$ . Notice that it belongs to  $(S_{x\hat{\sigma}})_\alpha$ .

We want to show that  $g$  is nowhere zero as a section of  $\mathcal{O}(\alpha)$ . Polynomial  $g \in S_{x\hat{\sigma}}$  is invertible, with inverse  $g^{-1} \in (S_{x\hat{\sigma}})_{-\alpha}$ . But then  $g^{-1} \cdot g = 1 \in (S_{x\hat{\sigma}})_0$ . If  $g$  were zero at some point  $p \in X_\Sigma$ , then we would have  $0 = g^{-1}(p) \cdot g(p) = 1$ , a contradiction.

The fact that  $g \in S_{x\hat{\sigma}}$  is nowhere zero on  $\text{Spec } S_{x\hat{\sigma}}$  (as a polynomial) is a consequence of  $g$  being invertible in this ring. Now we can set  $\bar{f} = g^{-1}f$  and then  $\bar{f}$  is a regular function on  $\text{Spec}(S_{x\hat{\sigma}})_0$ . We need to see if  $\bar{f}(p) = 0$  is equivalent to  $\bar{f}(\lambda_1, \dots, \lambda_r) = 0$ . In fact, even more is true:  $\bar{f}(p) = \bar{f}(\lambda_1, \dots, \lambda_r)$ . That follows from the fact that  $\bar{f}$  is a function both on  $\text{Spec } S_{x\hat{\sigma}}$  and on  $\text{Spec}(S_{x\hat{\sigma}})_0$  and its evaluation at any point is the same as at its image.  $\square$

**Corollary 2.5.** *Suppose  $\alpha \in \text{Pic } X_\Sigma$ . Suppose  $f_1, f_2 \in S_\alpha$  are polynomials and  $s_1, s_2$  are the corresponding sections of  $\mathcal{O}(\alpha)$ . Also fix, as above,  $p \in X_\Sigma$  and  $(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$  such that  $\pi(\lambda_1, \dots, \lambda_r) = p$ . Then if  $f_2(\lambda_1, \dots, \lambda_r)$  and  $s_2(p)$  are non-zero, we get*

$$\frac{f_1(\lambda_1, \dots, \lambda_r)}{f_2(\lambda_1, \dots, \lambda_r)} = \frac{s_1(p)}{s_2(p)}.$$

*Proof.* Take  $\mu \in \mathbb{C}$  such that  $f_1(\lambda_1, \dots, \lambda_r) = \mu f_2(\lambda_1, \dots, \lambda_r)$ . Then use the previous fact for  $f_1 - \mu f_2$  and the corresponding section  $s_1 - \mu s_2$ .  $\square$

### 3. APOLARITY

Introduce  $T = \mathbb{C}[y_1, \dots, y_r]$ . We will think of  $T$  as an  $S$ -module, where the multiplication (denoted by  $\lrcorner$ ) is induced by

$$x_i \lrcorner y_1^{b_1} \cdot \dots \cdot y_r^{b_r} = \begin{cases} y_1^{b_1} \cdot \dots \cdot y_i^{b_i-1} \cdot \dots \cdot y_r^{b_r} & \text{if } b_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is called apolarity and when  $g \lrcorner F = 0$  we often say that  $g$  is apolar to  $F$ . The grading on  $T$  is the same as on  $S$ :

$$\deg y_i := [D_{\rho_i}].$$

*Remark 3.1.* Notice that here we changed the notation from  $\circ$  to  $\lrcorner$ , and we gave up multiplying by the constant  $b_i$ . So the multiplication is not the same as in the introduction. But it would be the same if we somehow identified  $y_i^b$  from the introduction with  $b! \cdot y_i^b$  from this section. This can be done by taking  $T$  to be the ring of divided powers, see [IK99, Appendix A], or [Eis95, Chapter A2.4] for a coordinate free version. For characteristic zero, this amounts to setting  $y_i^{(b)} = \frac{y_i^b}{b!}$ . But here we do not need  $T$  to be a ring, we only need it to be a module. So we might as well write  $y_i^b$  instead of  $y_i^{(b)}$ . It will not matter, provided we do not multiply  $y_i^{b_1}$  by  $y_i^{b_2}$ . This will make some calculations easier.

*Remark 3.2.* Notice that when we take  $g \in S_\alpha$  and  $F \in T_\beta$ , then  $g \lrcorner F$  is of degree  $\alpha - \beta$  for any  $\alpha, \beta \in \text{Cl } X_\Sigma$ . That follows from the fact that when we multiply by subsequent  $x_i$ 's, the degree of  $F$  decreases by  $[D_{\rho_i}]$ . This means that, although  $T$  is not a graded  $S$ -module, it becomes a graded  $S$ -module if we define the grading by

$$\deg y_i = -[D_{\rho_i}].$$

Futhermore, if  $F \in T$  is homogeneous, we will denote by  $F^\perp$  its annihilator, which is a homogeneous ideal in that case.

From now on assume  $X_\Sigma$  is a proper toric variety. Then we have  $S_0 = T_0 = \mathbb{C}$  and  $S_\alpha, T_\alpha$  are finite-dimensional vector spaces for any  $\alpha \in \text{Cl } X_\Sigma$ .

**Proposition 3.3.** *The map  $S_\alpha \times T_\alpha \rightarrow S_0 = \mathbb{C}$  given by  $(g, F) \mapsto g \lrcorner F$  is a duality for any  $\alpha \in \text{Cl } X_\Sigma$ .*

*Proof.* We will show that the basis

$$\{x_1^{a_1} \cdots x_r^{a_r} : [a_1 D_{\rho_1} + \cdots + a_r D_{\rho_r}] = \alpha\}$$

is dual to

$$\{y_1^{b_1} \cdots y_r^{b_r} : [b_1 D_{\rho_1} + \cdots + b_r D_{\rho_r}] = \alpha\}.$$

We know that  $x_1^{a_1} \cdots x_r^{a_r} \lrcorner y_1^{a_1} \cdots y_r^{a_r} = 1$ . Consider the value of  $x_1^{a_1} \cdots x_r^{a_r} \lrcorner y_1^{b_1} \cdots y_r^{b_r}$  when  $(a_1, \dots, a_r) \neq (b_1, \dots, b_r)$ . We know that

$$(4) \quad x_1^{a_1} \cdots x_r^{a_r} \lrcorner y_1^{b_1} \cdots y_r^{b_r} = \begin{cases} y_1^{b_1 - a_1} \cdots y_r^{b_r - a_r} & \text{if } b_i \geq a_i \text{ for all } i. \\ 0 & \text{otherwise.} \end{cases}$$

We want to prove (4) is zero, so suppose otherwise. The degree of (4) is zero. But the only monomial whose degree is the trivial class is the constant monomial 1 (we are using that  $X_\Sigma$  is proper). This implies that  $b_i = a_i$  for all  $i$ . But this cannot be true, since we assumed  $(a_1, \dots, a_r) \neq (b_1, \dots, b_r)$ . This contradiction means that  $x_1^{a_1} \cdots x_r^{a_r} \lrcorner y_1^{b_1} \cdots y_r^{b_r} = 0$ , as desired.  $\square$

As a corollary, we see that  $T = \bigoplus_{\alpha \in \text{Cl } X_\Sigma} H^0(X, \mathcal{O}(\alpha))^*$ .

Combining Proposition 3.3 and Corollary 2.5, we get

**Proposition 3.4.** *For any  $\alpha \in \text{Pic } X_\Sigma$  such that  $\mathcal{O}(\alpha)$  is basepoint free, the map*

$$\varphi: X_\Sigma \rightarrow \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$$

*is given by*

$$(5) \quad \varphi([\lambda_1, \dots, \lambda_r]) = \left[ \sum_{\substack{b_1, \dots, b_r \in \mathbb{Z}_{\geq 0}: \\ y_1^{b_1} \cdots y_r^{b_r} \in T_\alpha}} \lambda_1^{b_1} \cdots \lambda_r^{b_r} \cdot y_1^{b_1} \cdots y_r^{b_r} \right].$$

*Proof.* In general, if  $\{s_i : i \in I\}$  is a basis of  $H^0(X, \mathcal{O}(\alpha))$  ( $I$  is some finite index set), and  $\{s^i : i \in I\} \subseteq H^0(X, \mathcal{O}(\alpha))^*$  is the dual basis, then

$$\varphi(p) = \left[ \sum_{i \in I} s_i(p) \cdot s^i \right],$$

where  $s_i(p)$  means evaluating section  $s_i$  at point  $p$ . Note that it does not make sense to talk about the value of a section in  $\mathbb{C}$ , but the quotient  $s_i(p)/s_j(p) \in \mathbb{C}$  makes sense, and the sum makes sense as a class in the projectivization of  $H^0(X, \mathcal{O}(\alpha))^*$ .

By the proof of Proposition 3.3, the monomials  $y_1^{b_1} \dots y_r^{b_r} \in T_\alpha$  form a dual basis to  $x_1^{b_1} \dots x_r^{b_r}$ . So from Corollary 2.5 we know that for any  $i = (b_1, \dots, b_r)$ ,  $i' = (b'_1, \dots, b'_r)$  such that  $s_{i'}(p)$  is non-zero we have

$$\frac{s_i(p)}{s_{i'}(p)} = \frac{(x_1^{b_1} \dots x_r^{b_r})(p)}{(x_1^{b'_1} \dots x_r^{b'_r})(p)} = \frac{\lambda_1^{b_1} \dots \lambda_r^{b_r}}{\lambda_1^{b'_1} \dots \lambda_r^{b'_r}}.$$

The formula (5) follows.  $\square$

**3.1. Hilbert function.** Fix  $F \in T_\alpha$ . The ring  $S/F^\perp$  is called the apolar ring of  $F$ . It is graded by the class group of  $X_\Sigma$ . Let us denote it by  $A_F$ . Consider its Hilbert function  $H : \text{Cl } X_\Sigma \rightarrow \mathbb{Z}_{\geq 0}$  given by

$$\beta \mapsto \dim_{\mathbb{C}}((A_F)_\beta).$$

The Hilbert function is symmetric. The proof for projective space also applies to toric varieties:

**Proposition 3.5.** *For any  $\beta \in \text{Cl } X_\Sigma$ :*

$$\dim_{\mathbb{C}}(A_F)_\beta = \dim_{\mathbb{C}}(A_F)_{\alpha-\beta}.$$

*Proof.* We will prove that the bilinear map  $(A_F)_\beta \times (A_F)_{\alpha-\beta} \rightarrow \mathbb{C} \cong (A_F)_0$  given by  $(g, h) \mapsto (g \cdot h) \lrcorner F$  is a duality. Take any  $g \in S_\beta$  such that  $g \lrcorner F \neq 0$ . Then there is  $h \in S_{\alpha-\beta}$  such that  $h \lrcorner (g \lrcorner F) \neq 0$  (because  $\lrcorner$  makes  $S_{\alpha-\beta}$  and  $T_{\alpha-\beta}$  dual by Proposition 3.3). But this means that  $(h \cdot g) \lrcorner F \neq 0$ . We have proven that multiplying by any non-zero  $g \in (A_F)_\beta$  is non-zero as a map  $(A_F)_{\alpha-\beta} \rightarrow \mathbb{C}$ . Similarly, multiplying by any non-zero  $h \in (A_F)_{\alpha-\beta}$  is non-zero as a map  $(A_F)_\beta \rightarrow \mathbb{C}$ . We are done.  $\square$

*Remark 3.6.* The values of the Hilbert function of  $S/F^\perp$  are the same as the ranks of the catalecticant homomorphisms. More precisely, let

$$C_F^\beta : S_\beta \rightarrow T_{\alpha-\beta}$$

be given by

$$g \mapsto g \lrcorner F.$$

This map is called the catalecticant homomorphism. We have

$$\text{rank } C_F^\beta = \dim_{\mathbb{C}}(A_F)_\beta.$$

This is because the graded piece of  $F^\perp$  of degree  $\beta$  is the kernel of  $C_F^\beta$ . For more on catalecticant homomorphisms, see [Tei14, Section 2] or [IK99, Chapter 1].

**3.2. Apolarity Lemma.** Let us work in a more general setting for a while.

Suppose  $X$  is a projective variety over  $\mathbb{C}$ . Let  $\mathcal{L}$  be a very ample line bundle on  $X$ , and  $\varphi: X \rightarrow \mathbb{P}(H^0(X, \mathcal{L})^*)$  the associated morphism. For a closed subscheme  $i: R \hookrightarrow X$ ,  $\langle R \rangle$  denotes its linear span in  $\mathbb{P}(H^0(X, \mathcal{L})^*)$ , and  $\mathcal{I}_R$  denotes its ideal sheaf on  $X$ . Recall that for any line bundle on  $X$ , the vector subspace  $H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \subseteq H^0(X, \mathcal{L})$  consists of the sections which pull back to zero on  $R$ .

Let  $(\cdot \lrcorner \cdot): H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L})^* \rightarrow \mathbb{C}$  denote the natural pairing (this agrees with the notation introduced in the beginning of Section 3). Now we are ready to formulate the Apolarity Lemma:

**Proposition 3.7** (Apolarity Lemma, general version). *Let  $F \in H^0(X, \mathcal{L})^*$  be a non-zero element. Then for any closed subscheme  $i: R \hookrightarrow X$  we have*

$$F \in \langle R \rangle \iff H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \lrcorner F = 0.$$

*Proof.* Take any  $s \in H^0(X, \mathcal{L})$ , let  $H_s$  be the corresponding hyperplane in  $H^0(X, \mathcal{L})^*$ . Then,

$$\langle R \rangle \subseteq H_s \iff i^*(s) = 0 \iff s \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

Below we identify sections  $s \in H^0(X, \mathcal{L})$  with hyperplanes  $H_s$  in  $H^0(X, \mathcal{L})^*$ . Then for any  $R$

$$\begin{aligned} F \in \langle R \rangle &\iff \forall_{s \in H^0(X, \mathcal{L})} (\langle R \rangle \subseteq H_s \implies F \in H_s) \\ &\iff \forall_{s \in H^0(X, \mathcal{L})} (s \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \implies F \in H_s) \\ &\iff \forall_{s \in H^0(X, \mathcal{L})} (s \in H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \implies s \lrcorner F = 0) \\ &\iff H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \lrcorner F = 0. \end{aligned}$$

□

**Proposition 3.8** (Apolarity Lemma, toric version). *Let  $\Sigma$  be a simplicial fan, and  $X_\Sigma$  the toric variety defined by it. Let  $\alpha \in \text{Pic } X_\Sigma$  be a very ample class and  $\varphi: X_\Sigma \hookrightarrow \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$  be the associated morphism. Fix a non-zero  $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$ . Then for any closed subscheme  $R \hookrightarrow X_\Sigma$  we have*

$$F \in \langle R \rangle \iff I(R) \subseteq F^\perp.$$

Recall that here  $I(R)$  is the ideal of  $R$  from Definition 2.2.

*Proof.* From Proposition 3.7 we know that  $F \in \langle R \rangle$  if and only if  $I(R)_\alpha \subseteq F_\alpha^\perp$ . It remains to prove that  $I(R)_\alpha \subseteq F_\alpha^\perp$  implies  $I(R) \subseteq F^\perp$ . Suppose  $I(R)_\alpha \subseteq F_\alpha^\perp$ . Take any  $g \in I(R)_\beta$  for some  $\beta \in \text{Cl } X_\Sigma$ . We want to show that  $g \lrcorner F = 0$ . We have  $S_{\alpha-\beta} \cdot g \subseteq I_\alpha$ , because  $g$  is in the ideal. This means that  $(S_{\alpha-\beta} \cdot g) \lrcorner F = 0$ , i.e.  $S_{\alpha-\beta} \lrcorner (g \lrcorner F) = 0$ . Now,  $g \lrcorner F$  is an element of  $T_{\alpha-\beta}$ , which is zero when multiplied by anything from  $S_{\alpha-\beta}$ , which is equal to  $T_{\alpha-\beta}^*$  by Proposition 3.3. It follows that  $g \lrcorner F$  is zero. □

*Remark 3.9.* By Proposition 2.3, we might have taken  $I(R)^{\text{sat}}$  instead of  $I(R)$  in the proposition above. By [Cox95, Theorem 3.7], we might have taken any  $B$ -saturated ideal defining  $R$ .

#### 4. CATALECTICANT BOUNDS

We prove lower bounds for various kinds of rank (called catalecticant bounds). They help us to calculate these ranks in Section 5. See [Tei14, Section 2] for a different viewpoint on these types of lower bounds in the cases of the Veronese variety, Segre-Veronese variety and general varieties.

**Proposition 4.1.** *Let  $X$  be a proper variety and  $R$  be a zero-dimensional subscheme of  $X$  with ideal sheaf  $\mathcal{I}_R$ . Then for any line bundle  $\mathcal{L}$*

$$\text{length } R \geq h^0(X, \mathcal{L}) - h^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

*Recall that for a zero-dimensional scheme  $R$  is  $\dim_{\mathbb{C}} H^0(R, \mathcal{O}_R)$ .*

*Proof.* We have an exact sequence

$$0 \rightarrow \mathcal{I}_R \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_R \rightarrow 0.$$

We tensor it with  $\mathcal{L}$ :

$$0 \rightarrow \mathcal{I}_R \otimes \mathcal{L} \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_R \rightarrow 0.$$

After taking global sections (which are left-exact), we get an exact sequence

$$0 \rightarrow H^0(X, \mathcal{I}_R \otimes \mathcal{L}) \rightarrow H^0(X, \mathcal{L}) \rightarrow H^0(R, \mathcal{L}|_R).$$

It follows that

$$h^0(R, \mathcal{L}|_R) \geq h^0(X, \mathcal{L}) - h^0(X, \mathcal{I}_R \otimes \mathcal{L}).$$

But on a zero-dimensional scheme, every line bundle trivializes. This means  $h^0(R, \mathcal{L}|_R) = h^0(R, \mathcal{O}_R)$ , which is the length of  $R$ .  $\square$

From now on, assume again that  $X = X_{\Sigma}$  is a projective simplicial toric variety. Let us fix a very ample class  $\alpha \in \text{Pic } X_{\Sigma}$ . Suppose  $\beta \in \text{Cl } X_{\Sigma}$ . The linear map  $\lrcorner : S_{\beta} \otimes T_{\alpha} \rightarrow T_{\alpha-\beta}$  can be seen as coming from the morphism

$$\mathcal{O}(\beta) \otimes \mathcal{O}(\alpha - \beta) \rightarrow \mathcal{O}(\alpha)$$

by taking multiplication of global sections:

$$H^0(X_{\Sigma}, \mathcal{O}(\beta)) \otimes H^0(X_{\Sigma}, \mathcal{O}(\alpha - \beta)) \rightarrow H^0(X_{\Sigma}, \mathcal{O}(\alpha))$$

and rearranging the terms:

$$H^0(X_{\Sigma}, \mathcal{O}(\beta)) \otimes H^0(X_{\Sigma}, \mathcal{O}(\alpha))^* \xrightarrow{\lrcorner} H^0(X_{\Sigma}, \mathcal{O}(\alpha - \beta))^*.$$

For any  $\gamma \in \text{Cl } X_{\Sigma}$  the space  $H^0(X_{\Sigma}, \mathcal{O}(\gamma))$  is  $S_{\gamma}$  and we identify  $T_{\gamma}$  with  $H^0(X_{\Sigma}, \mathcal{O}(\gamma))^*$  by Proposition 3.3. Notice that if we fix  $F \in H^0(X_{\Sigma}, \mathcal{O}(\alpha))^*$ , then the map above becomes the catalecticant homomorphism

$$C_F^{\beta} : H^0(X_{\Sigma}, \mathcal{O}(\beta)) \rightarrow H^0(X_{\Sigma}, \mathcal{O}(\alpha - \beta))^*$$

from Remark 3.6.

As a corollary of Proposition 4.1 and the Apolarity Lemma (Proposition 3.8), we get the catalecticant bound in the special case of line bundles.

**Corollary 4.2** (Catalecticant bound for cactus rank). *For any  $\beta \in \text{Pic } X_\Sigma$ , and any  $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$  we have*

$$\text{cr}(F) \geq \text{rank } C_F^\beta.$$

*Proof.* Take any zero-dimensional scheme  $R \hookrightarrow X_\Sigma$  such that  $F \in \langle R \rangle$ . Let  $I$  be any  $B$ -saturated ideal defining  $R$ . We have

$$\begin{aligned} \text{length } R &\geq h^0(X_\Sigma, \mathcal{O}(\beta)) - h^0(X_\Sigma, \mathcal{I}_R \otimes \mathcal{O}(\beta)) = \dim_{\mathbb{C}}(S/I)_\beta \\ &\geq \dim_{\mathbb{C}}(S/F^\perp)_\beta = \dim_{\mathbb{C}} \text{im } C_F^\beta, \end{aligned}$$

where the first inequality follows from Proposition 4.1, and the second from Proposition 3.8. We also used that  $I(R)$  agrees with any saturated ideal defining  $R$  in degrees coming from  $\text{Pic } X_\Sigma$ , see Remark 3.9, and the fact that values of the Hilbert function are ranks of catalecticant homomorphisms (Remark 3.6).  $\square$

The bound for cactus rank does not hold for classes  $\beta \notin \text{Pic } X_\Sigma$ . See Subsection 5.2 for an example. But the bound does hold for rank and  $\beta \in \text{Cl}(X)$ :

**Proposition 4.3** (Catalecticant bound for rank). *For any  $\beta \in \text{Cl } X_\Sigma$ , and any  $F \in H^0(X_\Sigma, \mathcal{O}(\alpha)^*)$  we have*

$$\text{r}(F) \geq \text{rank } C_F^\beta.$$

The following proof is an adaptation of [Tei14, the “surprisingly quick proof” after equation (8)].

*Proof.* For any  $\gamma \in \text{Cl } X_\Sigma$  and any  $(\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$ , define a polynomial in  $y_1, \dots, y_r$

$$\psi_\gamma(\lambda_1, \dots, \lambda_r) = \sum_{\substack{a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}: \\ y_1^{a_1} \dots y_r^{a_r} \in T_\gamma}} \lambda_1^{a_1} \dots \lambda_r^{a_r} \cdot y_1^{a_1} \dots y_r^{a_r}.$$

First we prove the formula

$$g \lrcorner \psi_\alpha(\lambda_1, \dots, \lambda_r) = g(\lambda_1, \dots, \lambda_r) \psi_{\alpha-\beta}(\lambda_1, \dots, \lambda_r)$$

for any  $g \in S_\beta$  (here  $g(\lambda_1, \dots, \lambda_r)$  means evaluating the polynomial  $g$  at the  $\lambda_i$ 's). The formula is linear in  $g$ , so we may assume  $g = x_1^{b_1} \dots x_r^{b_r}$ .

Let  $P$  be the set of all monomials  $y_1^{a_1} \dots y_r^{a_r}$  of degree  $\alpha$  such that  $g \lrcorner y_1^{a_1} \dots y_r^{a_r} \neq 0$  (i.e.  $a_i \geq b_i$  for all  $i$ ). Then the map  $g \lrcorner$  is a bijection from  $P$  onto the set of all monomials of degree  $\alpha - \beta$  in variables  $y_1, \dots, y_r$  (injectivity is clear; for surjectivity note that for any  $y_1^{a'_1} \dots y_r^{a'_r} \in T_{\alpha-\beta}$

the monomial  $y_1^{a'_1+b_1} \cdots y_r^{a'_r+b_r} \in T_\alpha$  is what we are looking for). It follows that

$$\begin{aligned} g \lrcorner \psi_\alpha(\lambda_1, \dots, \lambda_r) &= \sum_{\substack{a'_1, \dots, a'_r \in \mathbb{Z}_{\geq 0}: \\ y_1^{a'_1} \cdots y_r^{a'_r} \in T_{\alpha-\beta}}} \lambda_1^{a'_1+b_1} \cdots \lambda_r^{a'_r+b_r} y_1^{a'_1} \cdots y_r^{a'_r} \\ &= \lambda_1^{b_1} \cdots \lambda_r^{b_r} \cdot \sum_{\substack{a'_1, \dots, a'_r \in \mathbb{Z}_{\geq 0}: \\ y_1^{a'_1} \cdots y_r^{a'_r} \in T_{\alpha-\beta}}} \lambda_1^{a'_1} \cdots \lambda_r^{a'_r} y_1^{a'_1} \cdots y_r^{a'_r} \\ &= g(\lambda_1, \dots, \lambda_r) \psi_{\alpha-\beta}(\lambda_1, \dots, \lambda_r). \end{aligned}$$

Now we proceed to the proof of the catalecticant bound. Take  $F \in H^0(X, \mathcal{O}(\alpha))^*$ . Then  $\mathbf{r}(F)$  is the least  $l$  such that  $F = \psi_\alpha(\lambda^1) + \cdots + \psi_\alpha(\lambda^l)$  for some  $\lambda^1, \dots, \lambda^l \in \mathbb{C}^r$  (basically because  $\psi_\alpha$  agrees with  $\varphi|_{\mathcal{O}(\alpha)}$  from Proposition 3.4). We want to bound from above the dimension of the image of the map  $C_F^\beta : S_\beta \rightarrow T_{\alpha-\beta}$ ,  $g \mapsto g \lrcorner F$ . But for any  $g \in S_\beta$  we have

$$g \lrcorner F = g \lrcorner (\psi_\alpha(\lambda^1) + \cdots + \psi_\alpha(\lambda^l)) = g(\lambda^1) \cdot \psi_{\alpha-\beta}(\lambda^1) + \cdots + g(\lambda^l) \cdot \psi_{\alpha-\beta}(\lambda^l).$$

So for any  $g$  in the domain of the map, the image  $C_F^\beta(g)$  is in

$$\langle \psi_{\alpha-\beta}(\lambda^1), \dots, \psi_{\alpha-\beta}(\lambda^l) \rangle.$$

It follows that the rank of  $C_F^\beta$  is at most  $l$ .  $\square$

**Proposition 4.4.** *Fix  $\beta \in \text{Cl } X_\Sigma$ . Then for any  $l \in \mathbb{Z}_+$  the set of points  $F \in \mathbb{P}(H^0(X_\Sigma, \mathcal{O}(\alpha))^*)$  such that  $\text{rank}(C_F^\beta) \leq l$  is Zariski-closed.*

*Proof.* Pick a basis of  $H^0(X_\Sigma, \mathcal{O}(\beta))$  and a basis of  $H^0(X_\Sigma, \mathcal{O}(\alpha-\beta))$ . Then  $\lrcorner$  becomes a matrix with entries in  $H^0(X_\Sigma, \mathcal{O}(\alpha))$ . In order to get the rank of the map  $C_F^\beta = \cdot \lrcorner F$ , we evaluate the matrix at  $F \in H^0(X_\Sigma, \mathcal{O}(\alpha))^*$ . Hence the set of those  $F$ 's such that the rank of  $\cdot \lrcorner F$  is at most  $l$  is given by the vanishing of the  $(l+1)$ -th minors of the matrix. These minors are polynomials from  $\text{Sym}^\bullet H^0(X_\Sigma, \mathcal{O}(\alpha))$ . We are done.  $\square$

**Corollary 4.5** (Catalecticant bound for border rank). *For any  $\beta \in \text{Cl } X_\Sigma$  and any  $F \in T_\alpha$  we have*

$$\mathbf{r}(F) \geq \text{rank } C_F^\beta.$$

*Proof.* Let  $k = \mathbf{r}(F)$ . From Proposition 4.3 we know that

$$\sigma_k^0(X) = \{[G] \in \mathbb{P}T_\alpha : \mathbf{r}(G) \leq k\} \subseteq \{[G] \in \mathbb{P}T_\alpha : \text{rank } C_G^\beta \leq k\}.$$

Since the set on the right hand side is closed (Proposition 4.4), we have

$$\sigma_k(X) = \overline{\sigma_k^0(X)} \subseteq \{[G] \in \mathbb{P}T_\alpha : \text{rank } C_G^\beta \leq k\},$$

and the claim follows.  $\square$



## 5. EXAMPLES

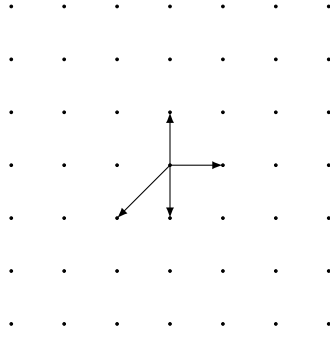
We use what we proved to look at some examples. In this section, we denote the coordinates of the ring  $S$  by Greek letters  $\alpha, \beta, \dots$  and the corresponding coordinates in  $T$  by  $x, y, \dots$  (possibly with subscripts).

We calculate ranks, cactus ranks and border ranks (denoted by  $\mathbf{r}(F)$ ,  $\mathbf{cr}(F)$ ,  $\mathbf{\underline{r}}(F)$ ) of some monomials  $F$  for toric surfaces embedded into projective spaces. See Definitions 1.1 and 1.4 for the definitions of these ranks.

**5.1. Hirzebruch surface  $\mathbb{F}_1$ .** Consider the set

$$\{\rho_{\alpha,0} = (1, 0), \rho_{\alpha,1} = (-1, -1), \rho_{\beta,0} = (0, 1), \rho_{\beta,1} = (0, -1)\}.$$

Let  $\Sigma$  be the only complete fan such that this set is the set of rays of  $\Sigma$ . The example in [CLS11, Example 3.1.16] is the same, only with a different ray arrangement. Then  $X_\Sigma$  is called the Hirzebruch surface  $\mathbb{F}_1$ . It is smooth.



Its class group is the free abelian group on two generators  $D_{\rho_{\alpha,0}} \sim D_{\rho_{\alpha,1}}$  and  $D_{\rho_{\beta,1}} \sim D_{\rho_{\beta,0}} + D_{\rho_{\alpha,0}}$ . Here and later in this section  $D_\rho$  is the toric invariant divisor corresponding to  $\rho$  (as in Section 2) and  $\sim$  means the linear equivalence. Let  $\alpha_0, \alpha_1, \beta_0, \beta_1$  be the variables corresponding to  $\rho_{\alpha,0}, \rho_{\alpha,1}, \rho_{\beta,0}, \rho_{\beta,1}$ . As a result, we may think of  $S$  as the polynomial ring  $\mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$  graded by  $\mathbb{Z}^2$ , where the grading is given by

$f$	$\alpha_0$	$\alpha_1$	$\beta_0$	$\beta_1$
$\deg f$	1	1	1	0
	0	0	1	1

The nef cone in  $(\text{Cl } X_\Sigma)_\mathbb{R}$  is generated by  $D_{\rho_{\alpha,0}}$  and  $D_{\rho_{\beta,0}} \sim D_{\rho_{\alpha,0}} + D_{\rho_{\beta,1}}$ . In this section, we will denote the value of  $\pi : \mathbb{C}^4 \setminus Z \rightarrow X_\Sigma$  by  $[\lambda_0, \lambda_1; \mu_0, \mu_1]$ .

**Example 5.1.** Consider the monomial  $F := x_0 x_1 y_0 y_1$ , where  $x_0, x_1, y_0, y_1$  is the basis dual to  $\alpha_0, \alpha_1, \beta_0, \beta_1$ . It has degree  $(3, 2)$ , so it is in the interior of the nef cone, so the corresponding line bundle is very ample. We claim that the rank and the cactus rank of  $F$  are four, and that the border rank is three:

$\mathbf{r}(F)$	$\mathbf{cr}(F)$	$\mathbf{\underline{r}}(F)$
4	4	3

Let us compute the Hilbert function of the apolar algebra of  $F$ .

	0	1	2	1		
	1	3	3	1		
	1	2	1	0		

Notice that it can only be non-zero in the first quadrant. Hence, the symmetry of the Hilbert function (see Proposition 3.5) implies it can only be non-zero in the rectangle with vertices  $(0, 0)$ ,  $(3, 0)$ ,  $(3, 2)$ ,  $(0, 2)$ . Computing each value of the Hilbert function is just computing the kernel of a linear map. For instance, for degree  $(1, 0)$ , we have

$$(a_0\alpha_0 + a_1\alpha_1) \lrcorner x_0x_1y_0y_1 = a_0x_1y_0y_1 + a_1x_0y_0y_1,$$

which is zero if and only if  $a_0 = 0$  and  $a_1 = 0$ . Hence, the Hilbert function is

$$\dim_{\mathbb{C}}(S/F^{\perp})_{(1,0)} = \dim_{\mathbb{C}}S_{(1,0)} - \dim_{\mathbb{C}}F^{\perp}_{(1,0)} = 2 - 0 = 2.$$

For degree  $(2, 1)$ , we get

$$(a\alpha_0^2\beta_1 + b\alpha_0\alpha_1\beta_1 + c\alpha_1^2\beta_1 + d\alpha_0\beta_0 + e\alpha_1\beta_0) \lrcorner x_0x_1y_0y_1 = by_0 + dx_1y_1 + ex_0y_1.$$

So the result is zero precisely for vectors of the form  $(a, 0, c, 0, 0)$ , where  $a, c \in \mathbb{C}$ . Then

$$\dim_{\mathbb{C}}(S/F^{\perp})_{(2,1)} = 5 - 2 = 3.$$

The apolar ideal  $F^{\perp}$  is  $(\alpha_0^2, \alpha_1^2, \beta_0^2, \beta_1^2)$ . (It is independent of the grading, so we can just copy the result from the Waring rank case, see [RS11].)

Firstly, we will show that the rank is at most four. By the Apolarity Lemma, toric version (Proposition 3.8), it is enough to find a reduced zero-dimensional subscheme of length four  $R$  of  $X_{\Sigma}$  (i.e. a set of four points in  $X_{\Sigma}$ ) such that  $I(R) \subseteq F^{\perp}$ . The subscheme defined by  $I = (\alpha_0^2 - \alpha_1^2, \beta_0^2 - \alpha_1^2\beta_1^2) \subseteq F^{\perp}$  satisfies these requirements. This scheme is a reduced union of four points:  $[1, 1; 1, 1]$ ,  $[1, 1; 1, -1]$ ,  $[1, -1; 1, 1]$ ,  $[1, -1; 1, -1]$ . As a consequence, we may write

$$x_0x_1y_0y_1 = \frac{1}{4}(\varphi(1, 1; 1, 1) - \varphi(1, 1; 1, -1) - \varphi(1, -1; 1, 1) + \varphi(1, -1; 1, -1)).$$

We will show that the cactus rank is at least four. Suppose it is at most three. Then there is a  $B$ -saturated homogeneous ideal  $I \subseteq F^{\perp}$  defining a

zero-dimensional subscheme  $R$  of length at most three. From the calculation of the Hilbert function, we know that  $\dim_{\mathbb{C}} F_{(2,1)}^{\perp} = 2$ . Let us calculate  $\dim_{\mathbb{C}} I_{(2,1)}$ . Since  $I$  is  $B$ -saturated, by Proposition 2.3, the vector subspace  $I_{(2,1)} \subseteq S_{(2,1)}$  are the sections which are zero on  $R$ . But from Proposition 4.1

$$3 \geq \text{length of } R \geq \dim_{\mathbb{C}} S_{(2,1)} - \dim_{\mathbb{C}} I_{(2,1)} = 5 - \dim_{\mathbb{C}} I_{(2,1)},$$

so

$$\dim_{\mathbb{C}} I_{(2,1)} \geq 2.$$

By the Apolarity Lemma (Proposition 3.8), we have  $I_{(2,1)} \subseteq (F^{\perp})_{(2,1)}$ . As the dimensions are equal, it follows that  $I_{(2,1)} = (F^{\perp})_{(2,1)}$ . This means  $\alpha_0^2 \beta_1, \alpha_1^2 \beta_1 \in I$ . But  $I$  is  $B$ -saturated, so  $\alpha_0 \alpha_1 \beta_1 \in I \subseteq F^{\perp}$ , which implies that  $\alpha_0 \alpha_1 \beta_1 \lrcorner x_0 x_1 y_0 y_1 = 0$ , a contradiction.

Let us show that border rank of  $F$  is at most three. Take  $p = [\lambda, 1; 1, \mu] \in \mathbb{F}_1$ . Then from Proposition 3.4, we know that

$$\begin{aligned} [\lambda, 1; 1, \mu] \mapsto & \lambda \mu \cdot \left( \lambda^2 \mu x_0^3 y_1^2 + \lambda \mu x_0^2 x_1 y_1^2 + \mu x_0 x_1^2 y_1^2 + \frac{\mu}{\lambda} x_1^3 y_1^2 \right. \\ & + \lambda x_0^2 y_0 y_1 + x_0 x_1 y_0 y_1 + \frac{1}{\lambda} x_1^2 y_0 y_1 \\ & \left. + \frac{1}{\mu} x_0 y_0^2 + \frac{1}{\mu \lambda} x_1 y_0^2 \right). \end{aligned}$$

But

$$[0, 1; 1, \mu] \mapsto \mu \cdot \left( \mu x_1^3 y_1^2 + x_1^2 y_0 y_1 + \frac{1}{\mu} x_1 y_0^2 \right),$$

and

$$[1, 0; 1, 0] \mapsto x_0 y_0^2.$$

Hence,

$$\begin{aligned} & -x_0 x_1 y_0 y_1 + \frac{1}{\lambda \mu} \varphi([\lambda, 1; 1, \mu]) - \frac{1}{\lambda \mu} \varphi([0, 1; 1, \mu]) - \frac{1}{\mu} \varphi([1, 0; 1, 0]) \\ & = \lambda^2 \mu x_0^3 y_1^2 + \lambda \mu x_0^2 x_1 y_1^2 + \mu x_0 x_1^2 y_1^2 + \lambda x_0 y_0^2 \xrightarrow{\lambda, \mu \rightarrow 0} 0. \end{aligned}$$

It follows that  $x_0 x_1 y_0 y_1$  is expressible as a limit of linear combinations of three points on  $X_{\Sigma}$ , so the border rank is at most three.

But there is another proof that the border rank of  $F$  is at most three. We will show that the third secant variety  $\sigma_3(X) = \mathbb{P}^8$ . It suffices to show that  $\dim \sigma_3(X_{\Sigma})$  is eight. We will use Terracini's Lemma (Proposition 1.8).

Since  $X_{\Sigma} \rightarrow \mathbb{P}(H^0(X_{\Sigma}, \mathcal{O}(\alpha))^*)$  is given by a parametrization, we can calculate the projectivized tangent space. Take points of the form  $[1, \lambda; \mu, 1]$ , where  $\lambda, \mu \in \mathbb{C}$ . Then

$$\varphi([1, \lambda; \mu, 1]) = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu \lambda, \mu \lambda^2, \mu^2, \mu^2 \lambda].$$

The coordinates are in the standard monomial basis of  $H^0(X_\Sigma, \mathcal{O}(\alpha))^*$ . The affine tangent space at  $\varphi([1, \lambda; \mu, 1])$  is spanned by the vector

$$v = [1, \lambda, \lambda^2, \lambda^3, \mu, \mu\lambda, \mu\lambda^2, \mu^2, \mu^2\lambda]$$

and its two derivatives with respect to  $\lambda$  and  $\mu$ :

$$\begin{aligned} \frac{\partial v}{\partial \lambda} &= [0, 1, 2\lambda, 3\lambda^2, 0, \mu, 2\mu\lambda, 0, \mu^2], \\ \frac{\partial v}{\partial \mu} &= [0, 0, 0, 0, 1, \lambda, \lambda^2, 2\mu, 2\mu\lambda], \end{aligned}$$

If we take three general points, say  $[1, x, y, 1], [1, s, t, 1], [1, u, v, 1]$ , we can look at the space spanned by the three tangent spaces. This will be the space spanned by the rows of the following matrix:

$$M = \begin{pmatrix} 1 & x & x^2 & x^3 & y & yx & yx^2 & y^2 & y^2x \\ 0 & 1 & 2x & 3x^2 & 0 & y & 2yx & 0 & y^2 \\ 0 & 0 & 0 & 0 & 1 & x & x^2 & 2y & 2yx \\ 1 & s & s^2 & s^3 & t & ts & ts^2 & t^2 & t^2s \\ 0 & 1 & 2s & 3s^2 & 0 & t & 2ts & 0 & t^2 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & 2t & 2ts \\ 1 & u & u^2 & u^3 & v & vu & vu^2 & v^2 & v^2u \\ 0 & 1 & 2u & 3u^2 & 0 & v & 2vu & 0 & v^2 \\ 0 & 0 & 0 & 0 & 1 & u & u^2 & 2v & 2vu \end{pmatrix}$$

We can calculate the determinant using for instance Macaulay2

$$\det M = (s - u)(u - x)(s - x)(ys - xt - yu + tu + xv - sv)^4.$$

This is non-zero for general points on the variety. This means that the tangent space of the cone of the third secant variety at a general point has dimension nine, so  $\dim \sigma_3(X_\Sigma) = 8$ , hence  $\sigma_3(X)$  fills the whole space.

Finally, the border rank is at least three by Corollary 4.5. We are using it for the class  $(2, 1)$ , recall that  $\dim_{\mathbb{C}}(S/F^\perp)_\beta = \text{rank } C_F^\beta$ .

*Remark 5.2.* We could also define the smoothable  $X$ -rank:

$$\text{sr}_X(F) = \min\{\text{length } R : R \hookrightarrow X, \dim R = 0, F \in \langle R \rangle, R \text{ smoothable}\}.$$

For the definition of a smoothable scheme, see [IK99, Definition 5.16]. For more on the smoothable rank, see [BB15]. We always have  $\text{cr}(F) \leq \text{sr}(F) \leq \text{r}(F)$ , so in the case of  $\mathbb{F}_1$  and  $F = x_0x_1y_0y_1$  we get  $\text{sr}(F) = 4$ . In particular, we obtain what the authors in [BB15] call a “wild” case, i.e. the border rank is strictly less than the smoothable rank.

**Example 5.3.** For a similar case on the same variety, let  $F = x_0^2x_1^2y_0y_1$ , then  $\deg F = (5, 2)$ . Here the line bundle  $\mathcal{O}(5, 2)$  gives an embedding of  $X_\Sigma$  into  $\mathbb{P}^{14}$ . We show that here the rank and the cactus rank are six, and that the border rank is five:

$\mathbf{r}(F)$	$\mathbf{cr}(F)$	$\mathbf{l}(F)$
6	6	5

The apolar ideal is  $F^\perp = (\alpha_0^3, \alpha_0^3, \beta_0^2, \beta_1^2)$ . The Hilbert function of  $S/F^\perp$  is the following:

	0	1	2	3	2	1		
	1	3	5	5	3	1		
	1	2	3	2	1	0		

The ideal  $I = (\alpha_0^3 - \alpha_1^3, \beta_0^2 - \beta_1^2 \alpha_1^2) \subseteq F^\perp$  is a  $B$ -saturated radical homogeneous ideal defining a subscheme of length six, so the rank is at most six.

Suppose there is a homogeneous  $B$ -saturated ideal  $I \subseteq F^\perp$  defining a subscheme of length five. We have

$$S_{(3,1)} = \langle \alpha_0^2 \beta_0, \alpha_0 \alpha_1 \beta_0, \alpha_1^2 \beta_0, \alpha_0^3 \beta_1, \alpha_0^2 \alpha_1 \beta_1, \alpha_0 \alpha_1^2 \beta_1, \alpha_1^3 \beta_1 \rangle, \text{ and}$$

$$(F^\perp)_{(3,1)} = \langle \alpha_0^3 \beta_1, \alpha_1^3 \beta_1 \rangle.$$

From Propostion 4.1 we have  $\dim_{\mathbb{C}}(S/I)_{(3,1)} \leq 5$ , so  $\dim_{\mathbb{C}} I_{(3,1)} \geq 7 - 5 = 2$ . But  $I_{(3,1)} \subseteq (F^\perp)_{(3,1)}$  from the Apolarity Lemma (Proposition 3.8), and also  $\dim_{\mathbb{C}}(F^\perp)_{(3,1)} = 2$ . This means that  $I_{(3,1)} = (F^\perp)_{(3,1)}$ .

Hence,  $\alpha_0^3 \beta_1, \alpha_1^3 \beta_1 \in I$ . As  $I$  is  $B$ -saturated, we get  $\alpha_0^2 \alpha_1^2 \beta_1 \in I \subseteq F^\perp$ , but this is a contradiction since  $\alpha_0^2 \alpha_1^2 \beta_1 \lrcorner F \neq 0$ .

The border rank is at least five because of Corollary 4.5. Similarly to what we did before, we show that fifth secant variety fills the whole space, so the border rank of any polynomial is at most five. Here  $\varphi = \varphi_{|\mathcal{O}(5,2)|}$  is given (in the standard monomial basis) by

$$[1, \lambda; \mu, 1] \mapsto [1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5, \mu, \lambda\mu, \lambda^2\mu, \lambda^3\mu, \lambda^4\mu, \mu^2, \lambda\mu^2, \lambda^2\mu^2, \lambda^3\mu^2].$$

The tangent space of the affine cone of  $X_\Sigma$  is spanned by  $v = \varphi(1, \lambda; \mu, 1)$  and the two derivatives

$$\begin{aligned}\frac{\partial v}{\partial \lambda} &= [0, 1, 2\lambda, 3\lambda^2, 4\lambda^3, 5\lambda^4, 0, \mu, 2\lambda\mu, 3\lambda^2\mu, 4\lambda^3\mu, 0, \mu^2, 2\lambda\mu^2, 3\lambda^2\mu^2], \\ \frac{\partial v}{\partial \mu} &= [0, 0, 0, 0, 0, 0, 1, \lambda, \lambda^2, \lambda^3, \lambda^4, 2\mu, 2\lambda\mu, 2\lambda^2\mu, 2\lambda^3\mu].\end{aligned}$$

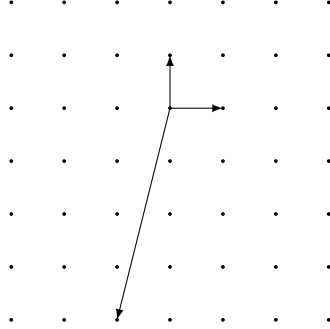
If we take five points, say  $[1, x; y, 1], [1, u; v, 1], [1, s; t, 1], [1, a, b, 1], [1, c, d, 1]$ , we get that the tangent space of the affine cone of  $\sigma_5(X_\Sigma)$  is spanned by the rows of the following matrix:

$$\begin{pmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 & y & xy & x^2y & x^3y & x^4y & y^2 & xy^2 & x^2y^2 & x^3y^2 \\ 0 & 1 & 2x & 3x^2 & 4x^3 & 5x^4 & 0 & y & 2xy & 3x^2y & 4x^3y & 0 & y^2 & 2xy^2 & 3x^2y^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & x^2 & x^3 & x^4 & 2y & 2xy & 2x^2y & 2x^3y \\ 1 & s & s^2 & s^3 & s^4 & s^5 & t & st & s^2t & s^3t & s^4t & t^2 & st^2 & s^2t^2 & s^3t^2 \\ 0 & 1 & 2s & 3s^2 & 4s^3 & 5s^4 & 0 & t & 2st & 3s^2t & 4s^3t & 0 & t^2 & 2st^2 & 3s^2t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s & s^2 & s^3 & s^4 & 2t & 2st & 2s^2t & 2s^3t \\ 1 & u & u^2 & u^3 & u^4 & u^5 & v & uv & u^2v & u^3v & u^4v & v^2 & uv^2 & u^2v^2 & u^3v^2 \\ 0 & 1 & 2u & 3u^2 & 4u^3 & 5u^4 & 0 & v & 2uv & 3u^2v & 4u^3v & 0 & v^2 & 2uv^2 & 3u^2v^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & u^2 & u^3 & u^4 & 2v & 2uv & 2u^2v & 2u^3v \\ 1 & a & a^2 & a^3 & a^4 & a^5 & b & ab & a^2b & a^3b & a^4b & b^2 & ab^2 & a^2b^2 & a^3b^2 \\ 0 & 1 & 2a & 3a^2 & 4a^3 & 5a^4 & 0 & b & 2ab & 3a^2b & 4a^3b & 0 & b^2 & 2ab^2 & 3a^2b^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2 & a^3 & a^4 & 2b & 2ab & 2a^2b & 2a^3b \\ 1 & c & c^2 & c^3 & c^4 & c^5 & d & cd & c^2d & c^3d & c^4d & d^2 & cd^2 & c^2d^2 & c^3d^2 \\ 0 & 1 & 2c & 3c^2 & 4c^3 & 5c^4 & 0 & d & 2cd & 3c^2d & 4c^3d & 0 & d^2 & 2cd^2 & 3c^2d^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & c & c^2 & c^3 & c^4 & 2d & 2cd & 2c^2d & 2c^3d \end{pmatrix}$$

If we set  $(x, y, s, t, u, v, a, b, c, d) = (1, 2, 3, 4, 5, 6, 7, 9, 0, 2)$  and calculate the determinant in the field  $\mathbb{Z}/101$ , we get 34, something non-zero. This means that the determinant calculated in  $\mathbb{C}$  is also non-zero at this point, so it is non-zero on a dense open subset. Hence by Terracini's lemma (Proposition 1.8) the dimension of the affine cone of  $\sigma_5(X_\Sigma)$  is fifteen. It follows that  $\sigma_5(X_\Sigma) = \mathbb{P}^{14}$ . Thus the border rank of  $F$  is five.

**5.2. Weighted projective plane  $\mathbb{P}(1, 1, 4)$ .** Consider a set of rays  $\{\rho_x = (-1, -4), \rho_y = (1, 0), \rho_z = (0, 1)\}$ . Let  $\Sigma$  be the complete fan determined by these rays. This is a fan of  $\mathbb{P}(1, 1, 4)$ , the weighted projective space with weights 1, 1, 4, see [CLS11, Section 2.0, Subsection Weighted Projective

Space; and Example 3.1.17].



The class group is  $\mathbb{Z}$ , generated by  $D_{\rho_x} \sim D_{\rho_y}$ , and we know that  $D_{\rho_z} \sim 4D_{\rho_x}$ . The Cox ring is  $\mathbb{C}[\alpha, \beta, \gamma]$ , where  $\alpha, \beta, \gamma$  correspond to  $\rho_x, \rho_y, \rho_z$ , and the degrees are given by the vector  $(1, 1, 4)$ . Let  $x, y, z$  denote the dual coordinates. The Picard group is generated by  $\mathcal{O}(4)$ . The only singular point is  $[0, 0, 1]$ .

Consider the embedding given by  $\mathcal{O}_{X_\Sigma}(4)$ , which is a line bundle. It maps  $X$  into  $\mathbb{P}^5$  (since there are six monomials of degree 4:  $x^4, x^3y, x^2y^2, xy^3, y^4, z$ ). We calculate various ranks of  $F = x^2y^2$ . The results are shown in the following table

$r(F)$	$\text{cr}(F)$	$\underline{r}(F)$
3	2	3

The Hilbert function of  $A_F$  is  $(1, 2, 3, 2, 1)$  (here the first element of the sequence corresponds to  $\mathcal{O}_{X_\Sigma}$ , the next to  $\mathcal{O}_{X_\Sigma}(1)$ , and so on). This means (by Corollary 4.5) that  $\underline{r}(F) \geq 3$ .

We know that  $F^\perp = (\alpha^3, \beta^3, \gamma)$ , since the annihilator remains the same if we change the grading. Let  $I = (\alpha^3, \beta^3) \subseteq F^\perp$ . We show that the length of the scheme  $R := V(I)$  is two. This will mean that  $\text{cr}(F) \leq 2$ . Since  $R$  is supported at the point  $[0, 0, 1]$ , we can look at it on the affine open  $U_\sigma$ , where  $\sigma = \text{Cone}(\rho_x, \rho_y)$ . After localizing  $S = \mathbb{C}[\alpha, \beta, \gamma]$  at  $\gamma$  and taking degree 0, we get the ring

$$\mathbb{C} \left[ \frac{\alpha^4}{\gamma}, \frac{\alpha^3\beta}{\gamma}, \frac{\alpha^2\beta^2}{\gamma}, \frac{\alpha\beta^3}{\gamma}, \frac{\beta^4}{\gamma} \right].$$

Ideal  $I$  becomes the ideal generated by  $\frac{\alpha^4}{\gamma}, \frac{\alpha^3\beta}{\gamma}, \frac{\alpha\beta^3}{\gamma}, \frac{\beta^4}{\gamma}$  in this ring, so the quotient is a two-dimensional vector space with basis  $1, \frac{\alpha^2\beta^2}{\gamma}$ . Hence the length of  $R$  is two.

But the cactus rank cannot be 1, since  $x^2y^2$  is not in the image of  $\varphi_{|\mathcal{O}(4)|}$  (see Proposition 3.4). It follows that  $\text{cr}(F) = 2$ .

Now consider the ideal  $I = (\alpha^3 - \beta^3, \gamma) \subseteq F^\perp$ . We show that the length of the scheme defined by  $I$  is three. Since  $I$  is radical, the scheme given by  $I$  is reduced, hence this will show that  $r(F) \leq 3$ , as desired. But  $I = (\alpha - \beta, \gamma) \cap (\alpha - \varepsilon\beta, \gamma) \cap (\alpha - \varepsilon^2\beta, \gamma)$ , where  $\varepsilon = \frac{-1 + \sqrt{3}i}{2}$ , so the scheme given by  $I$  is the reduced union of  $[1, 1, 0], [\varepsilon, 1, 0], [\varepsilon^2, 1, 0]$ .

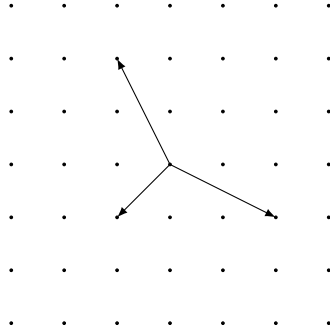
*Remark 5.4.* Since in this example

$$\text{rank } C_F^{\mathcal{O}(2)} = \dim(A_F)_2 = \dim(S/F^\perp)_2 = 3,$$

and  $\text{cr}(F) = 2$ , we see that the bound stated in point (1) of Proposition 1.7 does not hold for the cactus rank (and reflexive sheaves of rank one that are not line bundles).

*Remark 5.5.* One can also calculate that the projective tangent space in this embedding at the singular point  $[0, 0, 1]$  is the whole  $\mathbb{P}^5$  (this can be done by describing the embedding by a lattice polytope and calculating the Hilbert basis at a vertex corresponding to the singular point). It follows that the cactus rank of every point in  $\mathbb{P}^5$  is at most two, since any point of the tangent space at  $[0, 0, 1]$  can be reached by a linear span of a scheme of length two supported at  $[0, 0, 1]$ .

**5.3. Fake projective plane.** Consider a set of rays  $\{\rho_0 = (-1, -1), \rho_1 = (2, -1), \rho_2 = (-1, 2)\}$ . Let  $\Sigma$  be the complete fan determined by these rays. Then  $X_\Sigma$  is an example of a fake weighted projective space, see [Buc08, 6.2].



Let  $\alpha_0, \alpha_1, \alpha_2$  be the corresponding coordinates in  $S$ . The class group is generated by  $D_{\rho_0}, D_{\rho_1}, D_{\rho_2}$  with relations  $D_{\rho_0} \sim 2D_{\rho_1} - D_{\rho_2} \sim 2D_{\rho_2} - D_{\rho_1}$ . This is the same as a group with two generators  $D_{\rho_0}$  and  $D_{\rho_2} - D_{\rho_1}$  with the relation  $3(D_{\rho_2} - D_{\rho_1}) \sim 0$ . This choice gives an isomorphism with  $\mathbb{Z} \times \mathbb{Z}/3$  sending  $D_{\rho_0}$  to  $(1, 0)$  and  $D_{\rho_2} - D_{\rho_1}$  to  $(0, 1)$ . The Picard group is the subgroup generated by  $3D_{\rho_0}$ . It is free.

As a result,  $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$  is graded by  $\text{Cl } X_\Sigma = \mathbb{Z} \times \mathbb{Z}/3$ , where

$$\begin{aligned} \deg \alpha_0 &= (1, 0), \\ \deg \alpha_1 &= (1, 1), \\ \deg \alpha_2 &= (1, -1) = (1, 2), \end{aligned}$$

and  $\text{Pic } X_\Sigma$  is generated by  $(3, 0)$ .

The singular points are  $[1, 0, 0], [0, 1, 0], [0, 0, 1]$ .

Consider the line bundle  $\mathcal{O}(6, 0)$ . It is ample, because by [CLS11, Proposition 6.3.25] every proper toric surface is projective, and the line bundles  $\mathcal{O}(-3m, 0)$  for  $m < 0$  have no non-zero sections. By [CLS11, Proposition



6.1.10, (b)] it is very ample. It gives an embedding  $\varphi : X_\Sigma \hookrightarrow \mathbb{P}^9$ . We denote the dual coordinates by  $x_0, x_1, x_2$ .

**Example 5.6.** Let  $F = x_0^4 x_1 x_2 \in H^0(X_\Sigma, \mathcal{O}(6, 0))^*$ . The apolar ideal is  $(\alpha_0^5, \alpha_1^2, \alpha_2^2)$ . We claim that the cactus rank is two, the rank is at most five, and the border rank is at least two.

$\mathbf{r}(F)$	$\mathbf{cr}(F)$	$\mathbf{\underline{r}}(F)$
$\leq 5$	2	$\geq 2$

Note that  $F$  is not in the image of  $\varphi|_{\mathcal{O}(6,0)}$ , so the cactus rank and the border rank are at least two.

We show that the cactus rank is two. Consider the ideal  $I = (\alpha_1^2, \alpha_2^2) \subseteq F^\perp$ . It is saturated, since  $B$  in this case is  $(\alpha_0, \alpha_1, \alpha_2)$ , so it is the same as in the case of  $\mathbb{P}^2$ . We show that the length of the subscheme given by  $I$  is two. Since the support of the scheme is the point  $[1, 0, 0]$ , we check it on the set  $U_\sigma$ , where  $\sigma = \text{Cone}(\rho_1, \rho_2)$ . We localize with respect to  $\alpha_0$ , take degree zero, and get the ring

$$(6) \quad \mathbb{C} \left[ \frac{\alpha_1^3}{\alpha_0^3}, \frac{\alpha_2^3}{\alpha_0^3}, \frac{\alpha_1 \alpha_2}{\alpha_0^2} \right] \cong \mathbb{C}[u, v, w]/(w^3 - uv).$$

If we factor out by the ideal generated by  $\alpha_1^2$  and  $\alpha_2^2$ , we get

$$\mathbb{C}[u, v, w]/(w^3 - uv, u, v, w^2) \cong \mathbb{C}[w]/(w^2),$$

so the length of the scheme defined by  $I$  is two.

Now we show that the rank is at most five. Take a homogeneous ideal  $I = (\alpha_0^5 - \alpha_1^4 \alpha_2, \alpha_1^3 - \alpha_2^3) \subseteq F^\perp$ . We show that the length of the subscheme defined by  $I$  is five. From these equations we know that no coordinate can be zero, so we can check the length on the open subset  $U_\sigma$ , where  $\sigma = \text{Cone}(\rho_1, \rho_2)$ . We get the same ring as in Equation 6, and we want to factor it out by the ideal generated by  $\alpha_0^5 - \alpha_1^4 \alpha_2$  and  $\alpha_1^3 - \alpha_2^3$ . The second generator gives the relation  $u - v$ , and the first one the relation  $1 - vw$ . So we get the ring

$$\mathbb{C}[v, w]/(w^3 - v^2, 1 - vw).$$

But notice that  $1 = vw$  implies that  $w$  is non-zero. Hence

$$\begin{aligned} \mathbb{C}[v, w]/(w^3 - v^2, 1 - vw) &\cong \mathbb{C}[v, w, w^{-1}]/(w^3 - v^2, 1 - vw) \\ &\cong \mathbb{C}[v, w, w^{-1}]/(w^5 - 1, w^{-1} - v) \cong \mathbb{C}[w, w^{-1}]/(w^5 - 1). \end{aligned}$$

We get a reduced scheme of length five, so the rank is at most five.

*Remark 5.7.* It can be shown that  $\mathbf{\underline{r}}(F) \geq 3$ . To do this, it is enough to prove that the equations given by rank one reflexive sheaves  $\mathcal{O}(3, 0)$  and  $\mathcal{O}(3, 1)$  (given by minors of matrices as in the proof of Proposition 4.4) define the second secant variety. This can be done by computing that these equations define an irreducible variety of appropriate dimension over  $\mathbb{Q}$ . We do not provide details here. Finally, since  $F$  does not satisfy these equations, the claim follows.

**Example 5.8.** Now take  $F = x_0^2 x_1^2 x_2^2 \in H^0(X_\Sigma, \mathcal{O}(6, 0))^*$ . Here the apolar ideal is  $F^\perp = (\alpha_0^3, \alpha_1^3, \alpha_2^3)$ . We calculate the following

$\mathbf{r}(F)$	$\mathbf{cr}(F)$	$\mathbf{\underline{r}}(F)$
3	$\geq 2$	3

The cactus rank is at least two (because  $F$  is not in the image of  $\varphi_{|\mathcal{O}(6,0)|}$ ).

Let  $I = (\alpha_0^3 - \alpha_1^3, \alpha_1^3 - \alpha_2^3)$ . In this case also no coordinate can be zero, so we may calculate the length on  $U_\sigma$  (where  $\sigma$  is as before). We get the ring as in Equation 6 and the two generators become  $1 - u$  and  $u - v$ . So here the quotient ring is

$$\mathbb{C}[w]/(w^3 - 1).$$

This means that the rank is at most three (notice that we get a reduced scheme). We can calculate the Hilbert function of  $A_F = S/F^\perp$  (where  $F = x_0^2 x_1^2 x_2^2$ ). We have  $\dim_{\mathbb{C}}(A_F)_{(3,1)} = 3$ , so from Corollary 4.5 we get that  $\mathbf{\underline{r}}(F) \geq 3$ .

*Remark 5.9.* In fact, it can be shown that  $\mathbf{cr}(F) = 3$ . By looking at the polytope of the embedding by  $\mathcal{O}(6, 0)$  we can deduce that  $F$  is not in any of the projective tangent spaces at singular points. But the fact that  $\mathbf{\underline{r}}(F) \geq 3$  means that  $F$  is neither in any projective tangent space at a smooth point nor at any secant line passing through two points. We do not provide details here. It follows that  $\mathbf{cr}(F) > 2$ .

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MACIEJ GAŁAZKA, FACULTY OF MATHEMATICS, COMPUTER SCIENCE, AND MECHANICS,  
UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSZAWA, POLAND  
*E-mail address:* mgalazka@mimuw.edu.pl